



4-2

Standard Form of a Quadratic Function

Common Core State Standards

A-CED.A.2 Create equations in two or more variables . . . graph equations on coordinate axes with labels and scales. **Also F-IF.B.4, F-IF.B.6, F-IF.C.8, F-IF.C.9**
MP 1, MP 3, MP 4

Objective To graph quadratic functions written in standard form



What useful information does the graph give you? The equation?



SOLVE IT! **Getting Ready!**

The function $h = -0.01x^2 + 0.9x$ models the height h of the soccer ball as it travels distance x . What is the maximum height of the ball? Explain.

Lesson Vocabulary

- standard form

In Lesson 4-1, you worked with quadratic functions written in vertex form. Now you will use quadratic functions in *standard form*. The **standard form** of a quadratic function is $f(x) = ax^2 + bx + c$, where $a \neq 0$.

Essential Understanding For any quadratic function $f(x) = ax^2 + bx + c$, the values of a , b , and c provide key information about its graph.

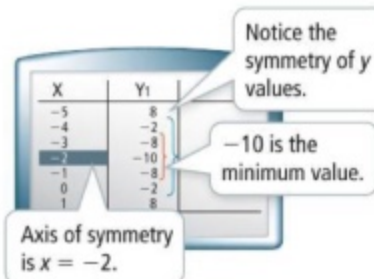
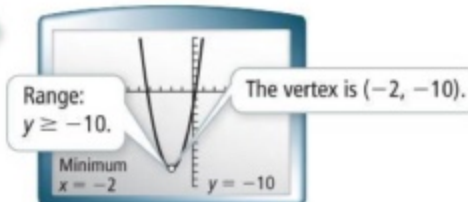
You can find information about the graph of a quadratic function (such as the vertex) easily from the vertex form. Such information is “hidden” in standard form. However, standard form is easier to enter into a graphing calculator.

Problem 1 Finding the Features of a Quadratic Function

Graphing Calculator What are the vertex, the axis of symmetry, the maximum or minimum value, and the range of $y = 2x^2 + 8x - 2$?

Plan

How can you use a calculator to find the features of a quadratic function in standard form? Graph the function. Then use the **CALC** and **TABLE** features.





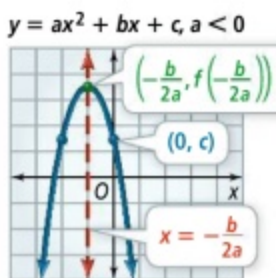
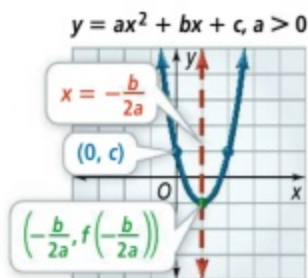
Got It? 1. What are the vertex, axis of symmetry, maximum or minimum value, and range of $y = -3x^2 - 4x + 6$?

You can find information about the quadratic function $f(x) = ax^2 + bx + c$ from the coefficients a and b , and from the constant term c .

Take note

Properties Quadratic Function in Standard Form

- The graph of $f(x) = ax^2 + bx + c$, $a \neq 0$, is a parabola.
- If $a > 0$, the parabola opens upward. If $a < 0$, the parabola opens downward.
- The axis of symmetry is the line $x = -\frac{b}{2a}$.
- The x -coordinate of the vertex is $-\frac{b}{2a}$. The y -coordinate of the vertex is the y -value of the function for $x = -\frac{b}{2a}$, or $y = f\left(-\frac{b}{2a}\right)$.
- The y -intercept is $(0, c)$.



Here's Why It Works You can expand the vertex form of a quadratic function to determine properties of the graph of a quadratic function written in standard form.

$$\begin{aligned} f(x) &= a(x - h)^2 + k \\ &= a(x^2 - 2hx + h^2) + k \\ &= ax^2 - 2ahx + ah^2 + k \\ &= ax^2 + (-2ah)x + (ah^2 + k) \end{aligned}$$

Compare to the standard form, $f(x) = ax^2 + bx + c$.

$$a = a \quad a \text{ in standard form is the same as } a \text{ in vertex form.}$$

$$b = -2ah$$

$$-\frac{b}{2a} = h \quad \text{Solve for } h.$$

Since, $h = -\frac{b}{2a}$, the axis of symmetry is $x = -\frac{b}{2a}$ and the vertex is

$$(h, k) = \left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right).$$



Problem 2 Graphing a Function of the Form $y = ax^2 + bx + c$

What is the graph of $y = x^2 + 2x + 3$?

Step 1 Identify a , b , and c .

$$a = 1, b = 2, c = 3$$

Step 2 The axis of symmetry is $x = -\frac{b}{2a}$.

$$x = -\frac{2}{2(1)}$$

Lightly sketch the line $x = -1$.

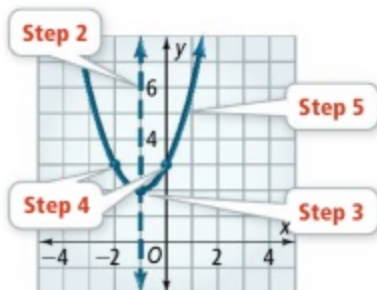
Step 3 The x -coordinate of the vertex is also $-\frac{b}{2a}$ or -1 .

$$\begin{aligned} \text{The } y\text{-coordinate is} \\ y = (-1)^2 + 2(-1) + 3 = 2. \end{aligned}$$

Plot the vertex $(-1, 2)$.

Step 4 Since $c = 3$, the y -intercept is $(0, 3)$. The reflection of $(0, 3)$ across $x = -1$ is $(-2, 3)$. Plot both points.

Step 5 $a > 0$ confirms that the graph opens upward. Draw a smooth curve through the points you found in Steps 3 and 4.



Think

How can you use the axis of symmetry?
The entire curve on one side of the axis is the mirror image of the curve on the other side.



Got It? 2. What is the graph of $y = -2x^2 + 2x - 5$?



Problem 3 Converting Standard Form to Vertex Form

What is the vertex form of $y = 2x^2 + 10x + 7$?

$$y = 2x^2 + 10x + 7$$

Identify a and b .

$$x = -\frac{b}{2a}$$

Find the x -coordinate of the vertex.

$$= -\frac{10}{2(2)}$$

$$= -2.5$$

$$y = 2(-2.5)^2 + 10(-2.5) + 7 \quad \text{Substitute } x = -2.5 \text{ into the equation.}$$

$$= -5.5$$

The vertex is $(-2.5, -5.5)$.

$$y = a(x - h)^2 + k$$

Write the vertex form.

$$y = 2[x - (-2.5)]^2 + (-5.5) \quad \text{Substitute } a = 2, h = -2.5, k = -5.5.$$

$$y = 2(x + 2.5)^2 - 5.5 \quad \text{Simplify.}$$

The vertex form is $y = 2(x + 2.5)^2 - 5.5$.



Got It? 3. What is the vertex form of $y = -x^2 + 4x - 5$?



Plan

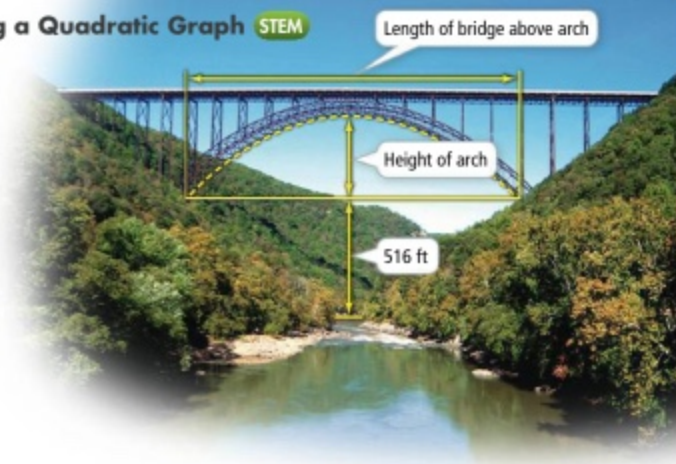
How do you find h , k , and a ?
Find the vertex. This gives you h and k . The value for a is the same in both forms.





Problem 4 Interpreting a Quadratic Graph STEM

Bridges The New River Gorge Bridge in West Virginia is the world's largest steel single arch bridge. You can model the arch with the function $y = -0.000498x^2 + 0.847x$, where x and y are in feet. How high above the river is the arch? How long is the section of bridge above the arch?



Know

A function that models the arch and the vertical distance from the base of the supports to the water

Need

The height of the arch above the support base and the length of the bridge above the arch

Plan

Find the vertex. The y -coordinate is the height of the arch above the support base. The x -coordinate is half the distance between the supports.

Think

How can you tell that the quadratic function has a maximum value? Since $a < 0$, the graph of the function opens down. The function has a maximum value.

Step 1 Find the vertex of the arch.

$$x = -\frac{b}{2a} = -\frac{0.847}{2(-0.000498)} \approx 850$$

$$y = -0.000498(850)^2 + 0.847(850) \approx 360$$

The vertex is about $(850, 360)$.

Step 2 Find the height of the arch above its supports.

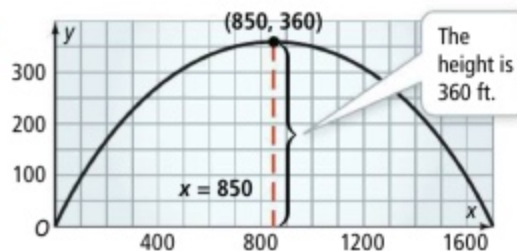
The y -coordinate of the vertex is the height of the arch above its supports. The arch is about 360 ft above its supports.

Step 3 Find the height of the arch above the river.

The arch is about $360 \text{ ft} + 516 \text{ ft} = 876 \text{ ft}$ above the river.

Step 4 Find the length of the bridge above the arch.

The x -coordinate of the vertex is half the length of the bridge above the arch. The length of that part of the bridge is about $850 \text{ ft} + 850 \text{ ft} = 1700 \text{ ft}$ long.



Got It? 4. a. The Zhaozhou Bridge in China is the one of the oldest known arch bridges, dating to A.D. 605. You can model the support arch with the function $f(x) = -0.001075x^2 + 0.131148x$, where x and y are measured in feet. How high is the arch above its supports?

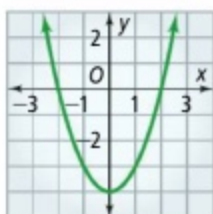
b. **Reasoning** Why does the model in part (a) not have a constant term?



Lesson Check

Do you know HOW?

1. Identify the vertex, axis of symmetry, and the maximum or minimum value of the parabola at the right.



Graph each function.

2. $y = x^2 - 2x + 4$
3. $y = -x^2 - 3x + 6$

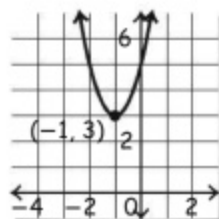
Write each function in vertex form.

4. $y = x^2 - 2x + 9$
5. $y = -x^2 + 3x - 1$

Do you UNDERSTAND? MATHEMATICAL PRACTICES

6. **Error Analysis** A student graphed the function $y = 2x^2 - 4x - 3$. Find and correct the error.

$$\begin{aligned} x &= \frac{-4}{2(2)} = -1 \\ y &= 2(-1)^2 - 4(-1) - 3 \\ &= 2 + 4 - 3 \\ &= 3 \\ \text{vertex } &(-1, 3) \end{aligned}$$



7. **Compare and Contrast** Explain the difference between finding the vertex of a function written in vertex form and finding the vertex of a function written in standard form.



Practice and Problem-Solving Exercises MATHEMATICAL PRACTICES

A Practice

Identify the vertex, the axis of symmetry, the maximum or minimum value, and the range of each parabola.

See Problem 1.

- | | | |
|-------------------------|-------------------------|--------------------------|
| 8. $y = x^2 + 2x + 1$ | 9. $y = -x^2 + 2x + 1$ | 10. $y = x^2 + 4x + 1$ |
| 11. $y = -x^2 + 2x + 5$ | 12. $y = 3x^2 - 4x - 2$ | 13. $y = -2x^2 - 3x + 4$ |
| 14. $y = 2x^2 - 6x + 3$ | 15. $y = -x^2 - x$ | 16. $y = 2x^2 + 5$ |

Graph each function.

See Problem 2.

- | | | |
|---------------------------|-----------------------------------|------------------------------------|
| 17. $y = x^2 + 6x + 9$ | 18. $y = -x^2 - 3x + 6$ | 19. $y = 2x^2 + 4x$ |
| 20. $y = 4x^2 - 12x + 9$ | 21. $y = -6x^2 - 12x - 1$ | 22. $y = -\frac{3}{4}x^2 + 6x + 6$ |
| 23. $y = 3x^2 - 12x + 10$ | 24. $y = \frac{1}{2}x^2 + 2x - 8$ | 25. $y = -4x^2 - 24x - 36$ |

Write each function in vertex form.

See Problem 3.

- | | |
|--------------------------|-----------------------------------|
| 26. $y = x^2 - 4x + 6$ | 27. $y = x^2 + 2x + 5$ |
| 28. $y = 4x^2 + 7x$ | 29. $y = 2x^2 - 5x + 12$ |
| 30. $y = -2x^2 + 8x + 3$ | 31. $y = \frac{9}{4}x^2 + 3x - 1$ |

32. **Economics** A model for a company's revenue from selling a software package is $R = -2.5p^2 + 500p$, where p is the price in dollars of the software. What price will maximize revenue? Find the maximum revenue.

See Problem 4.

B Apply

Sketch each parabola using the given information.

33. vertex (3, 6), y-intercept 2

34. vertex (-1, -4), y-intercept 3

35. vertex (0, 5), point (1, -2)

36. vertex (2, 3), point (6, 9)

37. **Think About a Plan** Suppose you work for a packaging company and are designing a box that has a rectangular bottom with a perimeter of 36 cm. The box must be 4 cm high. What dimensions give the maximum volume?

- How can you model the volume of the box with a quadratic function?
- What information can you get from the function to find the maximum volume?

38. **Landscaping** A town is planning a playground. It wants to fence in a rectangular space using an existing wall. What is the greatest area it can fence in using 100 ft of donated fencing?



For each function, the vertex of the function's graph is given. Find the unknown coefficients.

39. $y = x^2 + bx + c$; (3, -4)

40. $y = -3x^2 + bx + c$; (1, 0)

41. $y = ax^2 + 10x + c$; (-5, -27)

42. $y = c - ax^2 - 2x$; (-1, 3)

- STEM 43. **Physics** The height of a projectile fired straight up in the air with an initial velocity of 64 ft/s is $h = 64t - 16t^2$, where h is height in feet and t is time in seconds. The table represents the data for another projectile. Which projectile goes higher? How much higher?

Time (t)	Height (h)
0.5	20
1	32
1.5	36
2	32

44. A student says that the graph of $y = ax^2 + bx + c$ gets wider as a increases.
- Error Analysis** Use examples to show that the student is wrong.
 - Writing** Summarize the relationship between $|a|$ and the width of the graph of $y = ax^2 + bx + c$.

For each function, find the y-intercept.

45. $y = (x - 1)^2 + 2$

46. $y = -3(x + 2)^2 - 4$

47. $y = -\frac{2}{3}(x - 9)^2$

48. Use the functions $f(x) = 4x + 3$ and $g(x) = \frac{1}{2}x^2 + 2$ to answer parts (a)–(c).
- Which function has a greater rate of change from $x = 0$ to $x = 1$?
 - Which function has a greater rate of change from $x = 2$ and $x = 3$?
 - Does $g(x)$ ever have a greater rate of change than $f(x)$? Explain.



Challenge For each function, the vertex of the function's graph is given. Find a and b .

49. $y = ax^2 + bx - 27$; $(2, -3)$

50. $y = ax^2 + bx + 5$; $(-1, 4)$

51. $y = ax^2 + bx + 8$; $(2, -4)$

52. $y = ax^2 + bx$; $(-3, 2)$

53. Sketch the parabola with an axis of symmetry $x = 2$, y -intercept 1, and point $(3, 2.5)$.



Apply What You've Learned



Look back at the information about the sandwich shops and at the table of data on page 193.

- Victor plans to use the other shop owner's data to determine the relationship between the number of bags sold, x , and the selling price, s . Does the other shop owner's data set appear to be linear? Explain.
- Write an equation that models the data in the table. This is Victor's selling-price function.

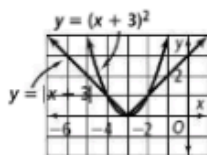
You can use the following relationship to construct a function that models Victor's profit from selling x bags of chips.

$$\text{Profit} = \text{Revenue} - \text{Cost}$$

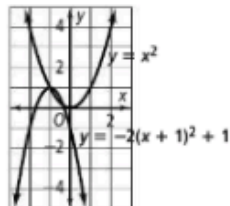
$$P(x) = R(x) - C(x)$$

- Write an equation for Victor's cost function $C(x)$.
- Victor's revenue will be the selling price times the number of bags sold. Use your result from part (b) to write and simplify an equation for Victor's revenue function $R(x)$.
- Use your results from parts (c) and (d) to write an equation for Victor's profit function $P(x)$.
- You have now written four functions. Which functions are linear functions and which are quadratic functions?

43. similar: same vertex $(-3, 0)$ and open upward, same domain (all real numbers), same range $(y \geq 0)$, same x -intercept, $(-3, 0)$; different: $y = |x + 3|$ is an absolute value function with y -intercept $(0, 3)$, and $y = (x + 3)^2$ is a quadratic function with y -intercept $(0, 9)$



45. stretch vertically by a factor of 2, reflect across the x -axis, translate 1 unit to the left and 1 unit up



47. a. 10; 9; -1 b. 9; 6; -3 c. 6; 1; -5
 d. The rate of change decreases at a greater rate.
 e. Yes; the rates of change increase as you move away from the origin on the negative x -axis.

49. $y = -7(x - 1)^2 + 2$

51. $y = -7x^2 + 5$

53. Answers may vary. Sample: $y = (x + 10)^2 - 4$

55. Answers may vary. Sample:
 $k = -2, a = 3; y = 3(x - 1)^2 - 2$

57. Answers may vary. Sample:
 $a = -6, k = 35; y = -6(x + 1)^2 + 35$

59. a. $ah^2 + k$ b. when $h = 0$

61. $y = \frac{1}{2}(x + 3)^2$ 63. $y = -\frac{1}{4}(x - 4)^2$

65. $y = -4(x + 3)^2$ 67. G

69. $2x + 2y = 225$

$$2y = 225 - 2x$$

$$y = \frac{225}{2} - x$$

$$A = xy = x\left(\frac{225}{2} - x\right)$$

$$A = -x^2 + 112.5x$$

Graph the function. There is a max. at about $(56.25, 3164.06)$, so the max. area is about 3164.06 ft^2 and the length of each side is 56.25 ft .

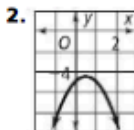
70. $(3, 2)$ 71. $(-10, 6)$ 72. $(1, 0, 3)$

73. $(0, 0)$ 74. $(-1, 0)$ 75. $(5, 0)$

Lesson 4-2

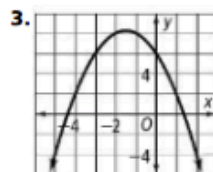
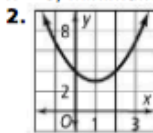
pp. 202-208

Got It? 1. vertex: $\left(-\frac{2}{3}, 7\frac{1}{3}\right)$; axis of symmetry: $x = -\frac{2}{3}$;
 maximum: $7\frac{1}{3}$; range: $y \leq 7\frac{1}{3}$



3. $y = -(x - 2)^2 - 1$ 4. a. 4 ft b. because the y -intercept is $(0, 0)$

Lesson Check 1. vertex: $(0, -4)$; axis of symmetry: $x = 0$; minimum: -4



4. $y = (x - 1)^2 + 8$ 5. $y = -\left(x - \frac{3}{2}\right)^2 + \frac{5}{4}$

6. Error in calculation of x . The correct calculation is:

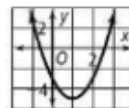
$$x = \frac{-(-4)}{2(2)} = 1$$

$$y = 2(1) - 4(1) - 3$$

$$= 2 - 4 - 3$$

$$= -5$$

Vertex: $(1, -5)$



7. The vertex of a function written in vertex form can easily be determined. It is (h, k) where $f(x) = a(x - h)^2 + k$. The vertex of a function in standard form is $\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$ where $f(x) = ax^2 + bx + c$.

Exercises 9. vertex: $(1, 2)$; axis of symmetry: $x = 1$;

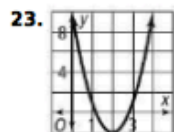
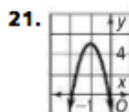
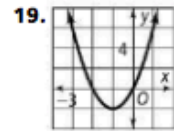
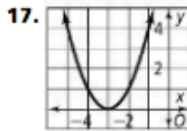
maximum: 2; range: $y \leq 2$ 11. vertex: $(1, 6)$; axis

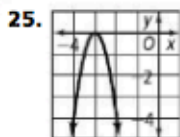
of symmetry: $x = 1$; maximum: 6; range: $y \leq 6$

13. vertex: $\left(-\frac{3}{4}, 5\frac{1}{8}\right)$; axis of symmetry: $x = -\frac{3}{4}$;

maximum: $5\frac{1}{8}$; range: $y \leq 5\frac{1}{8}$ 15. vertex: $\left(-\frac{1}{2}, \frac{1}{4}\right)$;

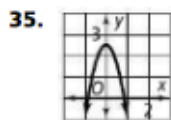
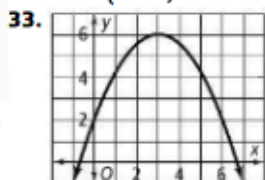
axis of symmetry: $x = -\frac{1}{2}$; maximum: $\frac{1}{4}$; range: $y \leq \frac{1}{4}$





27. $y = (x + 1)^2 + 4$ 29. $y = 2\left(x - \frac{5}{4}\right)^2 + \frac{71}{8}$

31. $y = \frac{9}{4}\left(x + \frac{2}{3}\right)^2 - 2$

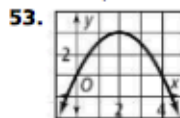


37. 4 cm \times 9 cm \times 9 cm 39. $b = -6$; $c = 5$

41. $a = 1$; $c = -2$ 43. a. The projectile represented by the equation goes higher by 28 feet. b. equation: $t = 0.27$ s and $t = 3.73$ s; table: $t = 0.38$ s and $t = 2.62$ s

45. (0, 3) 47. (0, -54) 49. $a = -6$, $b = 24$

51. $a = 3$, $b = -12$



Lesson 4-3

pp. 209-214

Got It? 1. $y = -3x^2 + x$ 2a. Rocket 2 b. Rocket 1: $D: 0 \leq t \leq 9.4$, $R: 0 \leq h \leq 352.6$; Rocket 2: $D: 0 \leq t \leq 12$, $R: 0 \leq h \leq 580$ c. The domains tell you how many seconds the rockets were in the air. The domains are different because the rockets were in the air for different amounts of time.

3. $y = -0.329x^2 + 9.798x + 15.571$; 88.5°F at 2:53 P.M. (although the meteorologist's prediction is 89° at 3 P.M.)

Lesson Check 1. $y = -2x^2 + 3x - 1$

2. $y = 2x^2 + 6x + 7.5$ 3. $y = -2x^2 + 10x - 13.5$

4. Answers may vary. Sample: A rough plot of the data will indicate whether the data are collinear (linear regression) or non-collinear where the data follows a curve (quadratic regression). 5. A parabola that opens up always attains greater values than one that opens down. 6. y is not a function of x since for one value of x , "3," there are 2 values of y , "4" and "0."

Exercises 7. $y = -x^2 + 3x - 4$ 9. $y = 2x^2 - x + 3$

11. $y = x^2 - 6x + 3$ 13. $y = x^2 + 2x$

15. $y = -x^2 + x - 2$ 17. a. $y = -16x^2 + 33x + 46$, where x is the number of seconds after release and y is the height in ft b. 28.5 ft c. about 63 ft 19. yes;

$y = -2x^2 + 3x + 5$ 21. yes; $y = 0.625x^2 - 1.75x + 1$

23. $y = 0.005x^2 - 1.95x + 120$; 66 mm

25. a. $y = -0.004x^2 + 0.859x + 27.53$ b. Answers may vary. Sample: domain: integers from -10 to 17 ; range: whole numbers from 18 to 42 c. the year 2003

d. The year 2022; the year is outside the domain of the data pts. 27. a. (3, 5) b. 3; substitute x - and y -values in the general form of a quadratic, then solve the resulting linear system for the coefficients.

Lesson 4-4

pp. 216-223

Got It? 1. a. $(x + 10)(x + 4)$ b. $(x - 5)(x - 6)$

c. $-(x + 2)(x - 16)$ 2. a. $7(n^2 - 3)$ b. $9(x + 2)(x - 1)$

c. $4(x^2 + 2x + 3)$ 3. a. $(x + 1)(4x + 3)$

b. $(x - 2)(2x - 3)$ c. No; $2x^2 + 2x + 2 = 2(x^2 + x + 1)$, there are no real factors of a and c whose product is 1 and whose sum is 1. 4. $(8x - 1)^2$

5. $(4x - 9)(4x + 9)$

Lesson Check 1. $(x + 4)(x + 2)$ 2. $(x - 12)(x - 1)$

3. $(x - 9)(x + 9)$ 4. $(5y - 6)(5y + 6)$ 5. $(y - 3)^2$

6. $(2x - 1)^2$ 7. $5x$ 8. $4a^2$ 9. 6 10. $7h$ 11. No; the middle term is not twice the product of the square root of the end terms. 12. For $a \neq 1$, look for two factors whose sum is b and whose product is ac . For $a = 1$, look for two factors whose sum is b and whose product is c .

13. $a^2 - 2ab + b^2 - 25$ Group the first 3 terms.

$$= (a^2 - 2ab + b^2) - 25$$

$$= (a - b)^2 - 5^2$$

$$= (a - b - 5)(a - b + 5)$$

Exercises 15. $(x + 2)(x + 3)$ 17. $(x + 2)(x + 8)$

19. $(x + 2)(x + 20)$ 21. $-(x - 1)(x - 12)$

23. $(x - 4)(x - 6)$ 25. $(x - 4)(x - 9)$

27. $-(x - 4)(x + 5)$ 29. $(c - 7)(c + 9)$

31. $-(t - 11)(t + 4)$ 33. $5b$; $5b(5b - 4)$

35. 5; $5(t + 1)(t - 2)$ 37. 9; $9(3p^2 - p + 2)$

39. $(x - 8)(2x - 3)$ 41. $(m - 3)(2m - 5)$

43. $(x - 12)(2x - 3)$ 45. $(y + 4)(5y - 8)$

47. $(x + 1)^2$ 49. $(k - 9)^2$

51. cannot be factored; 8 is not a perfect square and there are no positive factors of 32 that have a sum of 16.

53. $(x - 2)(x + 2)$ 55. cannot be factored; This is a sum of squares, not a difference of squares.

57. $(5x - 1)$ cm by $(5x - 1)$ cm 59. $2(3z + 2)(3z - 2)$

61. $16(2t + 1)(2t - 1)$ 63. $3(y + 5)(y + 3)$

65. $3(x + 1)(x - 9)$ 67. $2(x - 5)(2x - 1)$

69. $-\left(\frac{1}{4}s - 1\right)\left(\frac{1}{4}s + 1\right)$

71. The third line should be $x(2x - 5) - (2x - 5)$, and the final line should be $(x - 1)(2x - 5)$. 73. y ; $y(y - 1)$

75. 10; $10(x - 3)(x + 3)$ 77. 2; $2(x^2 - 37x + 6)$ 79. D

81. Answers may vary. 83. $(0.5t + 0.4)(0.5t - 0.4)$

85. $(x + 12)(x - 3)$ 87. $(2x + 9)(3x + 14)$

89. Factor the GCF, $4x^2$, from the terms to get $4x^2(x^2 + 6x + 8)$. Look for numbers whose product is 8 and whose sum is 6. The numbers 4 and 2 work. The complete factorization is $4x^2(x + 4)(x + 2)$.